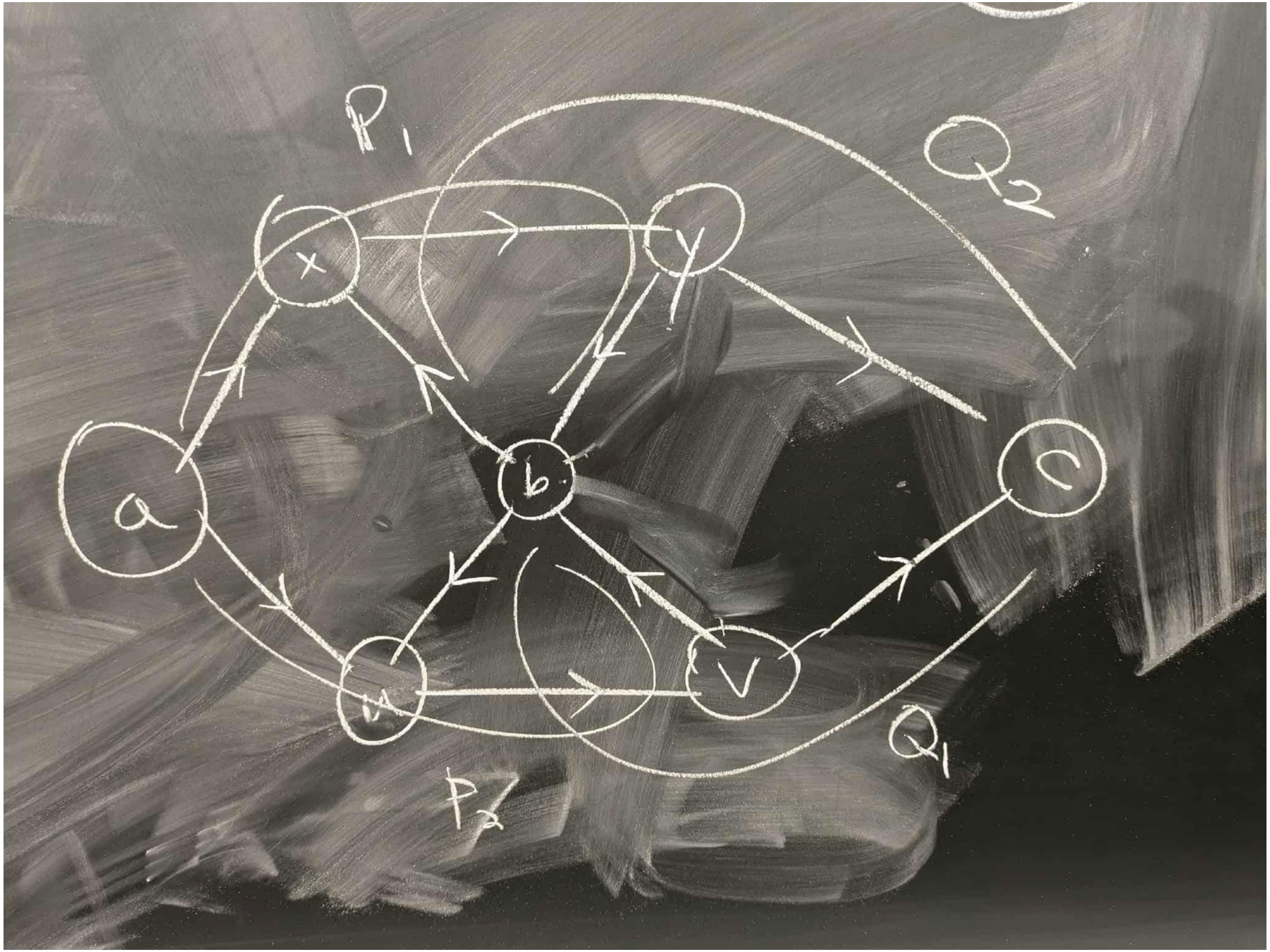
Advanced Algorithms

September 30, 2025

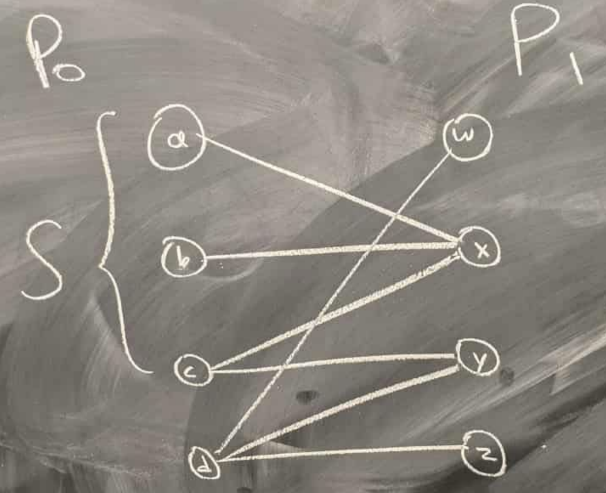
Logistics

- Assignment 1 done
  - Good job!
  - Grades published on gradescope
  - Will go through some things

- Exercise set 2
  - Due next Tuesday **in class.** Available now on course webpage
  - Opportunity for course feedback!



P1
Q2
x
y
a
b
c
u
v
Q1
P2



P_0 picks c.

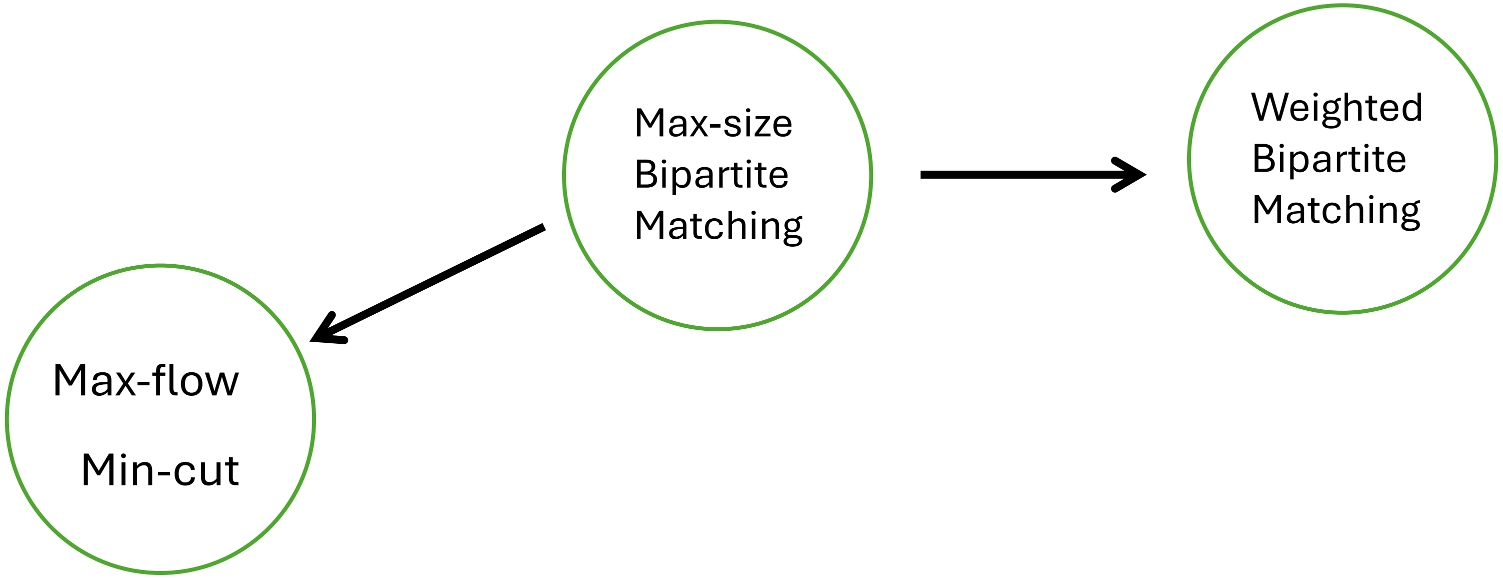
P_1 responds with y.

P_0 must choose d.

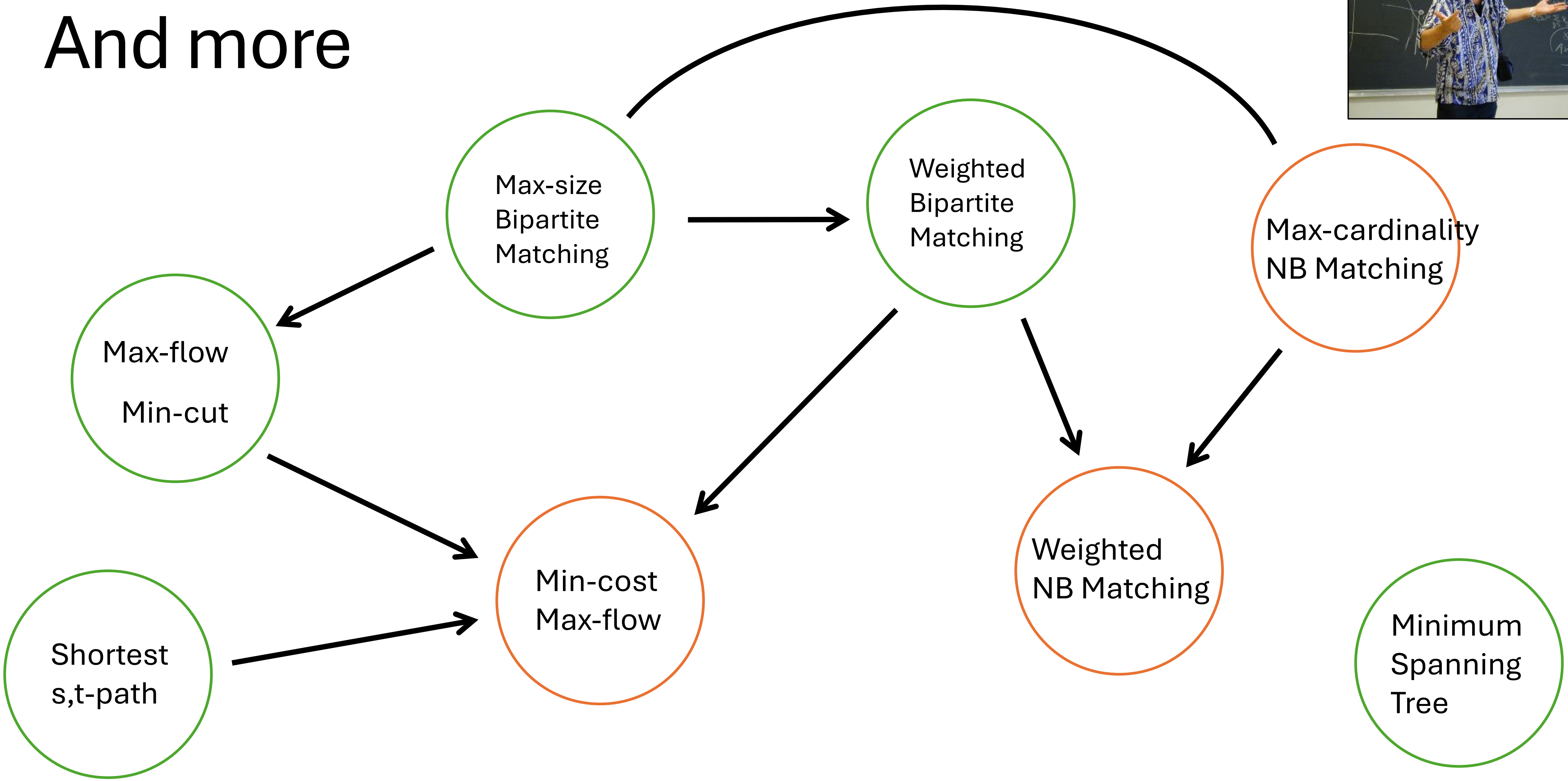
P_1 chooses z. (or w)

Win for P_1

Previously . . . on Advanced Algorithms



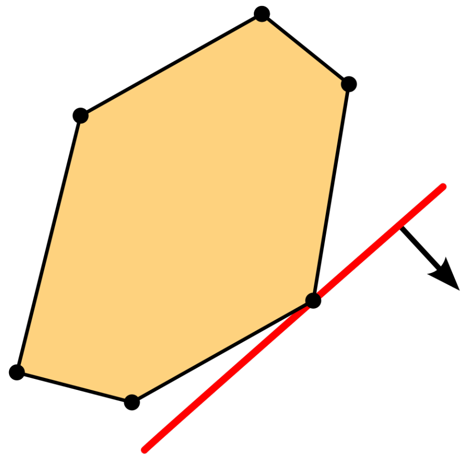
And more



Fundamental Problems in P

Up Next: Linear Programming

- A very general problem in P. Already models many problems
- Used as a subroutine in modern algorithm design
  - Approximation Algorithms
  - Online Algorithms
- An expressive language for all optimization problems



Example: Farmer

- A farmer is trying to decide which crops to plant on her 60 acres of land.

- She can plant either wheat or corn:
  - Each acre planted with wheat can be sold later for $200 profit
  - Each acre planted with corn yields $300 profit

- Planting requires labor and fertilizer:
  - Each wheat acre needs 3 hours of labor and 2 tons of fertilizer
  - Each corn acre needs 2 hours of labor and 4 tons of fertilizer

- She only has 100 hours and 120 tons of fertilizer available.

- How many acres of wheat and corn to plant if we want to maximize sales?

Farmer's LP

Maximize $200x + 300y$

Subject to

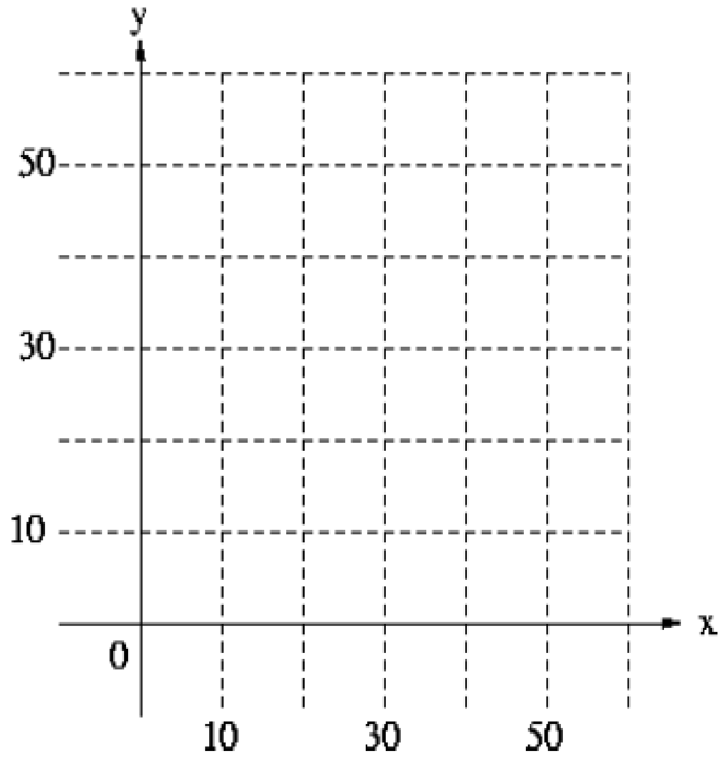
$3x + 2y \leq 100$	(hours of work available)
$2x + 4y \leq 120$	(fertilizer constraint)
$x + y \leq 60$	(can't plant too many acres)
$x \geq 0$	
$y \geq 0$	(non-negativity)

Realistic?

How to solve the LP?

- I will do this by hand . . . once
- (you will too on the Exercise set)

Feasible set



Non negative part of the plane

Max $200x + 300y$

s.t. $3x + 2y \leq 100$ (labor)

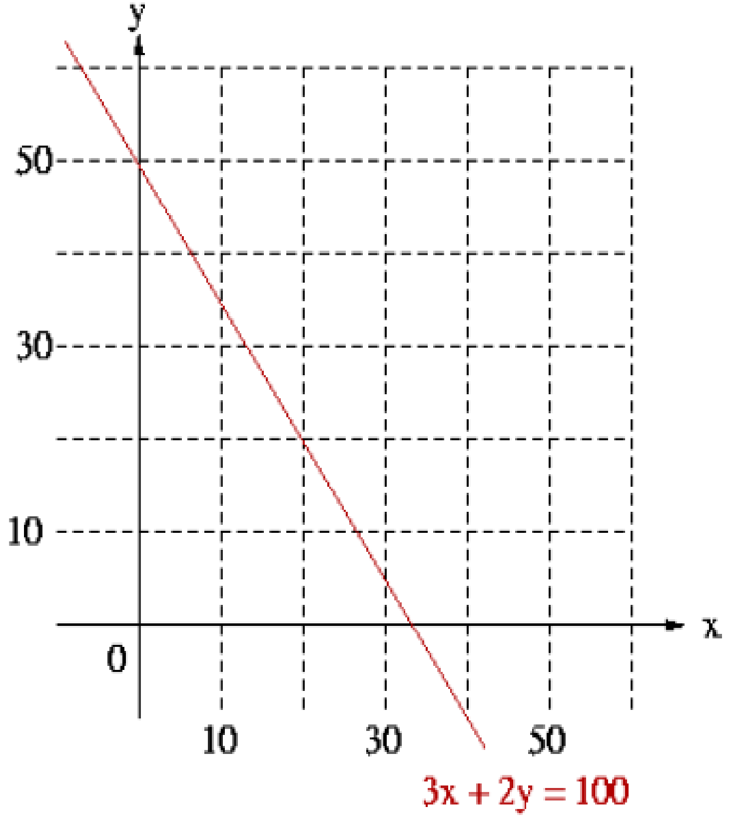
$2x + 4y \leq 120$ (fertilizer)

$x + y \leq 60$ (acres)

$x \geq 0$

$y \geq 0$

Feasible set

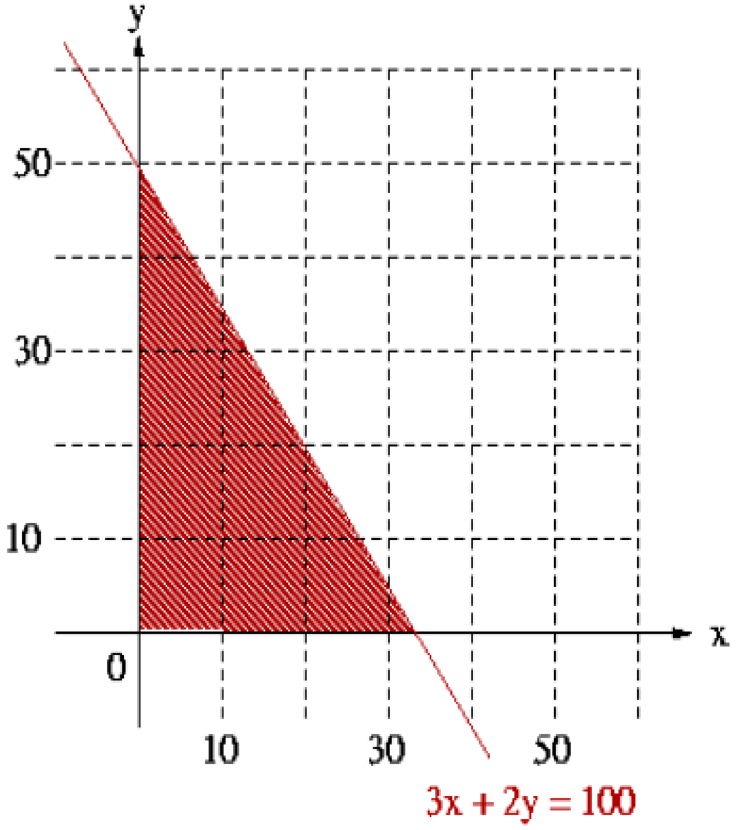


First constraint drawn as an equality

$$\text{Max } 200x + 300y$$

$$\begin{aligned} \text{s.t.} \quad & 3x + 2y \le 100 \text{ (labor)} \\ & 2x + 4y \le 120 \text{ (fertilizer)} \\ & x + y \le 60 \text{ (acres)} \\ & x \ge 0 \\ & y \ge 0 \end{aligned}$$

Feasible set



Feasible region for the first constraint

Max $200x + 300y$

s.t. $3x + 2y \leq 100$ (labor)

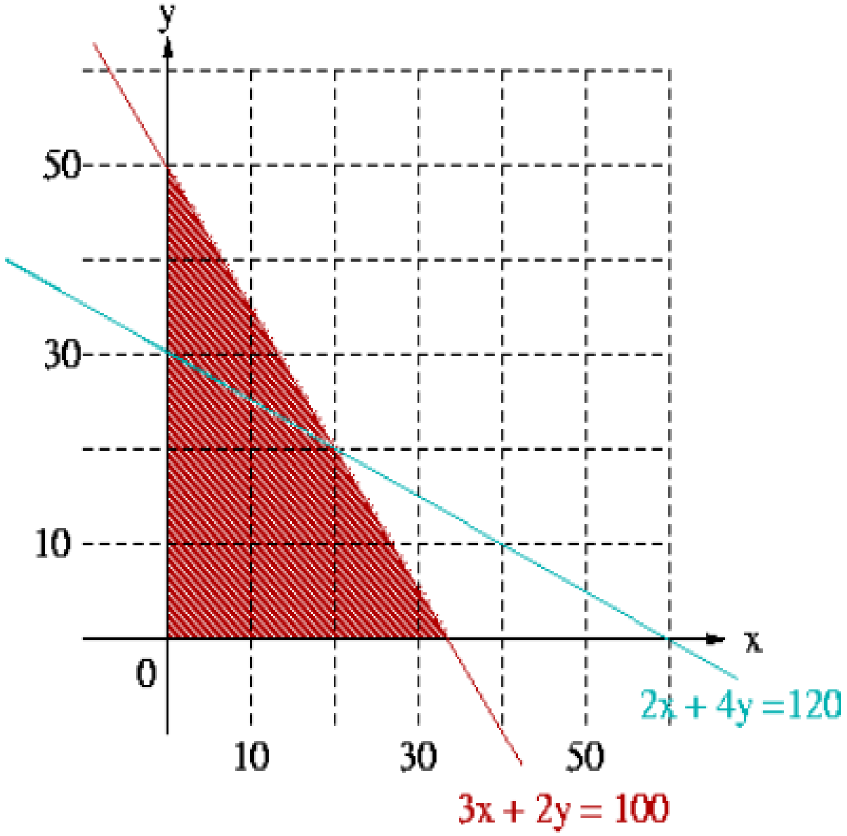
$2x + 4y \leq 120$ (fertilizer)

$x + y \leq 60$ (acres)

$x \geq 0$

$y \geq 0$

Feasible set



Second constraint drawn as an equality

Max 200x + 300y

s.t. 3x + 2y ≤ 100 (labor)

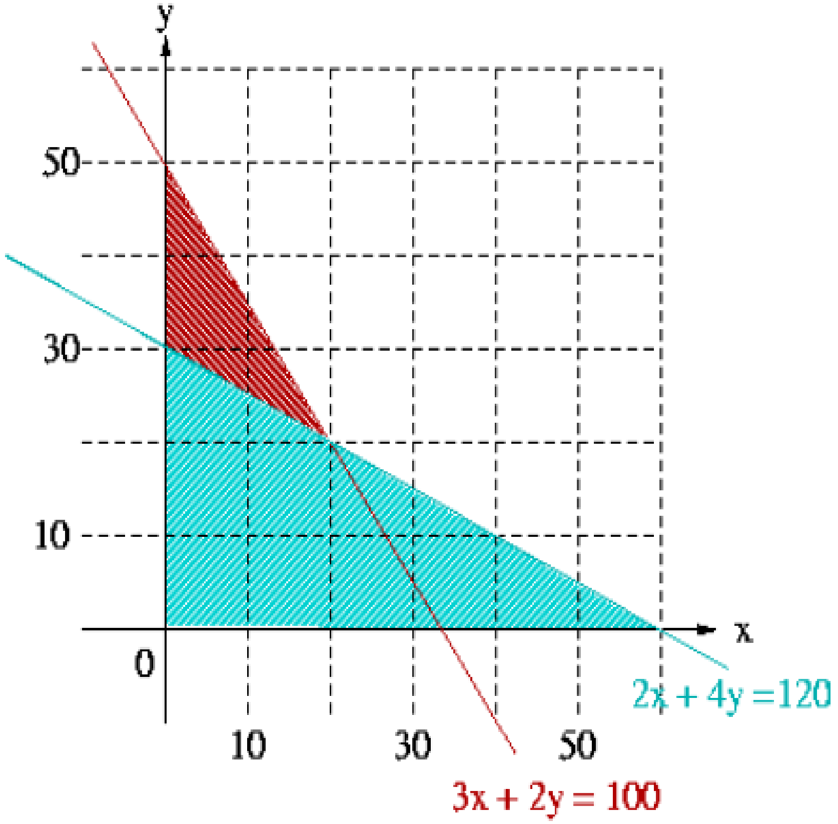
2x + 4y ≤ 120 (fertilizer)

x + y ≤ 60 (acres)

x ≥ 0

y ≥ 0

Feasible set



Feasible region for the second constraint

Max 200x + 300y

s.t. $3x + 2y \leq 100$ (labor)

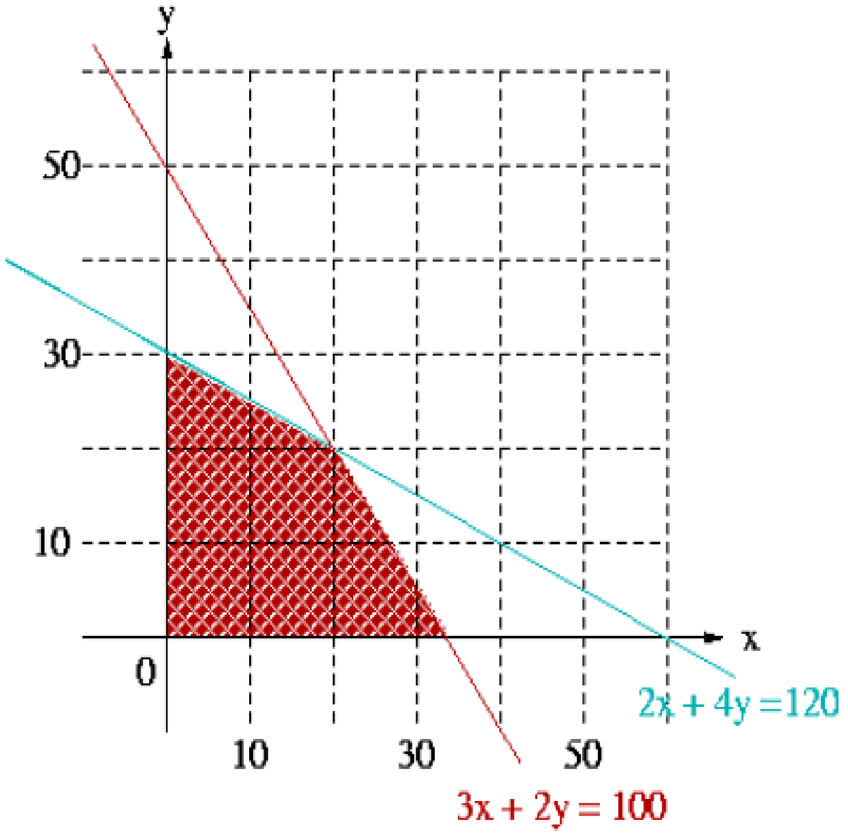
$2x + 4y \leq 120$ (fertilizer)

$x + y \leq 60$ (acres)

$x \geq 0$

$y \geq 0$

Feasible set



Intersection of the two feasible regions

Max $200x + 300y$

s.t. $3x + 2y \leq 100$ (labor)

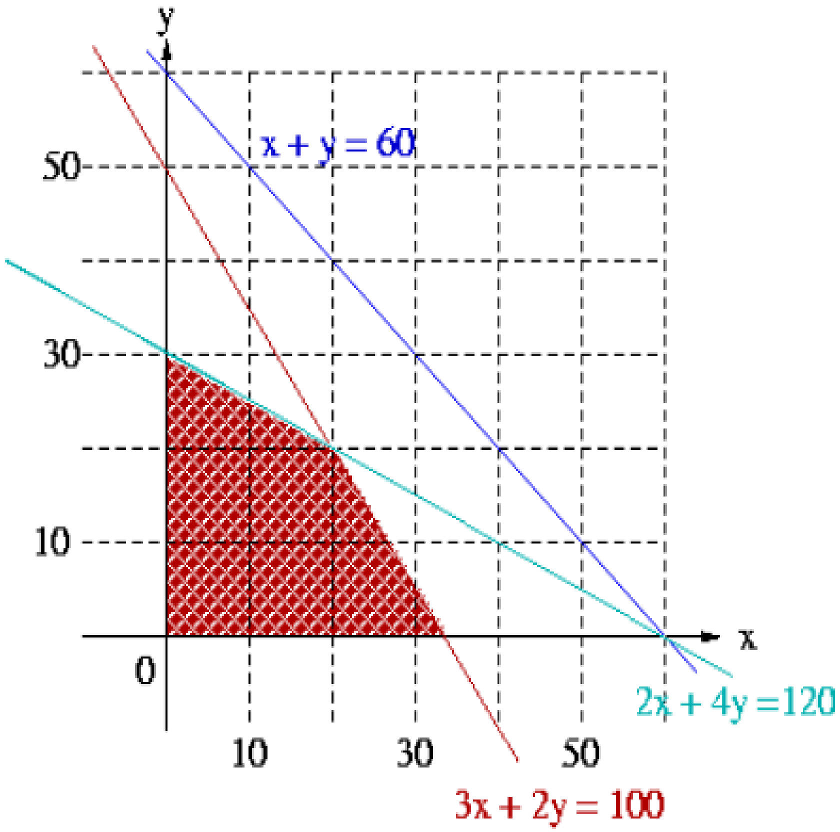
$2x + 4y \leq 120$ (fertilizer)

$x + y \leq 60$ (acres)

$x \geq 0$

$y \geq 0$

Feasible set



Third constraint drawn as an equality

Max 200x + 300y

s.t. $3x + 2y \leq 100$ (labor)

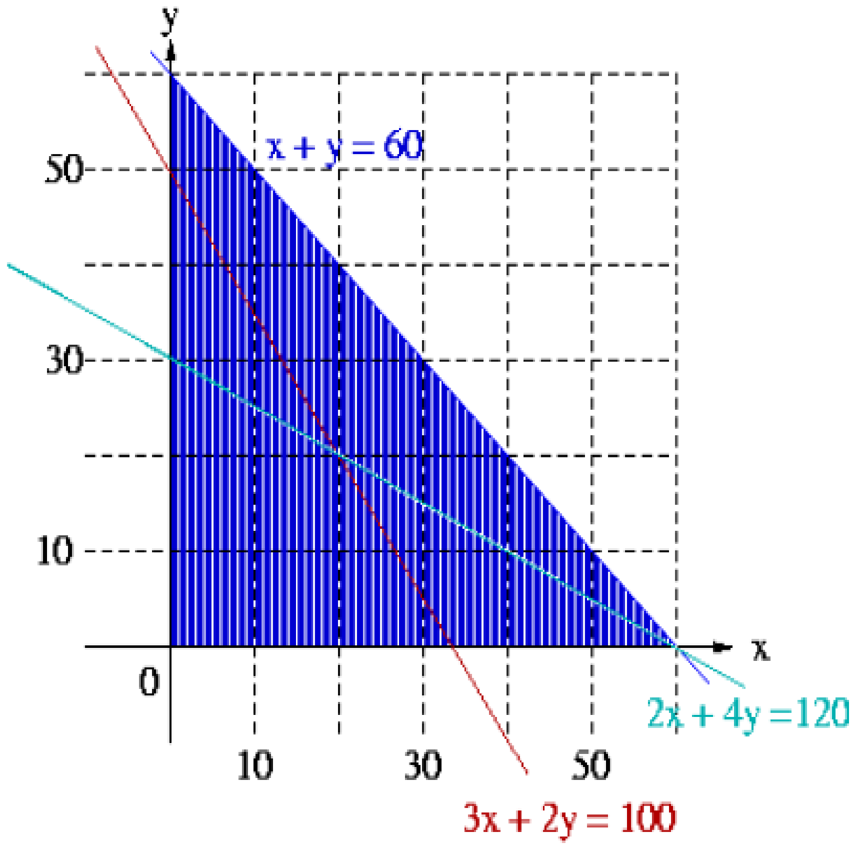
$2x + 4y \leq 120$ (fertilizer)

$x + y \leq 60$ (acres)

$x \geq 0$

$y \geq 0$

Feasible set



Feasible region for the third constraint

Max 200x + 300y

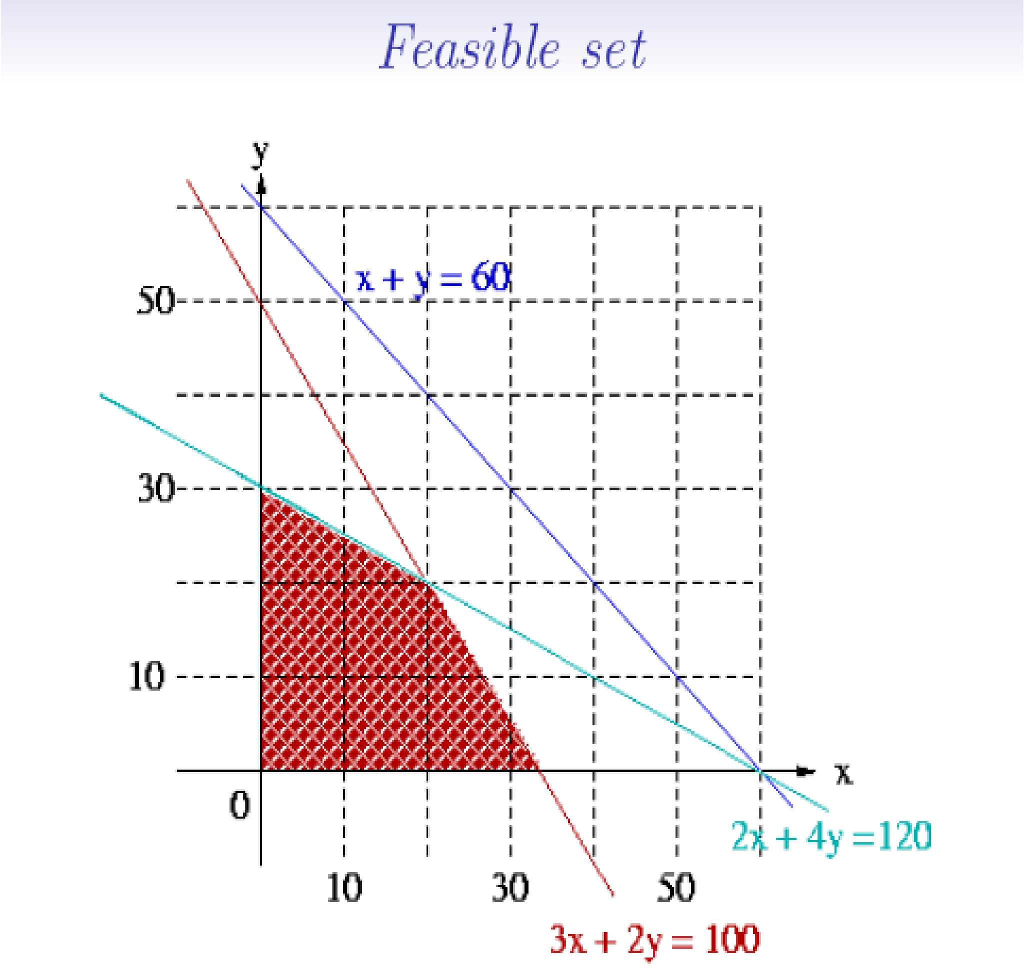
s.t. 3x + 2y ≤ 100 (labor)

2x + 4y ≤ 120 (fertilizer)

x + y ≤ 60 (acres)

x ≥ 0

y ≥ 0



Intersection of the three feasible regions

Max $200x + 300y$

s.t. $3x + 2y \le 100$ (labor)

$2x + 4y \le 120$ (fertilizer)

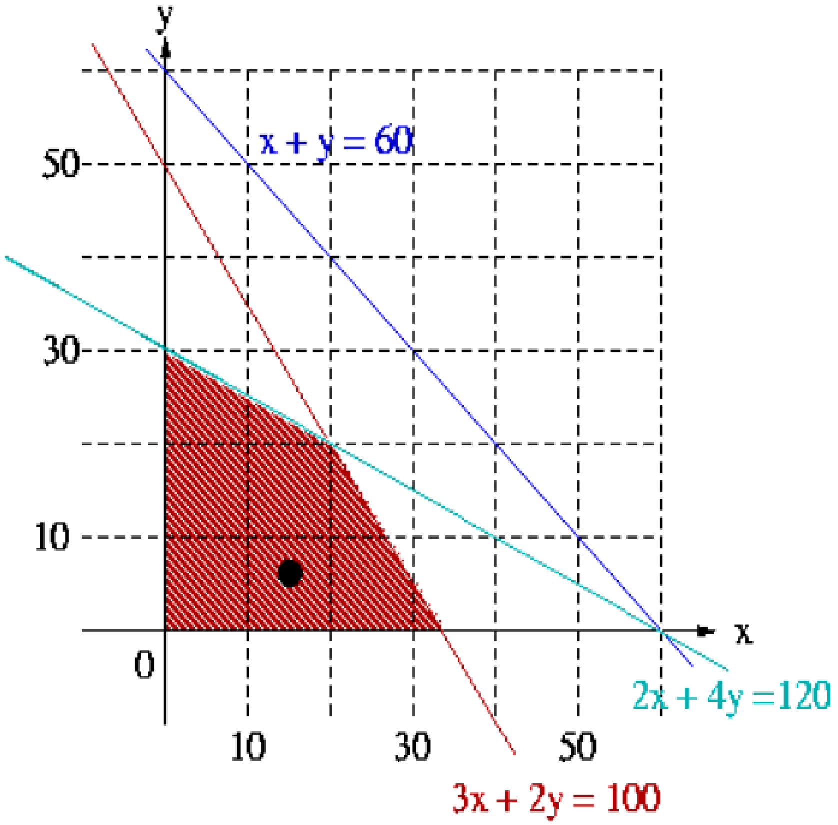
$x + y \le 60$ (acres)

$x \ge 0$

$y \ge 0$

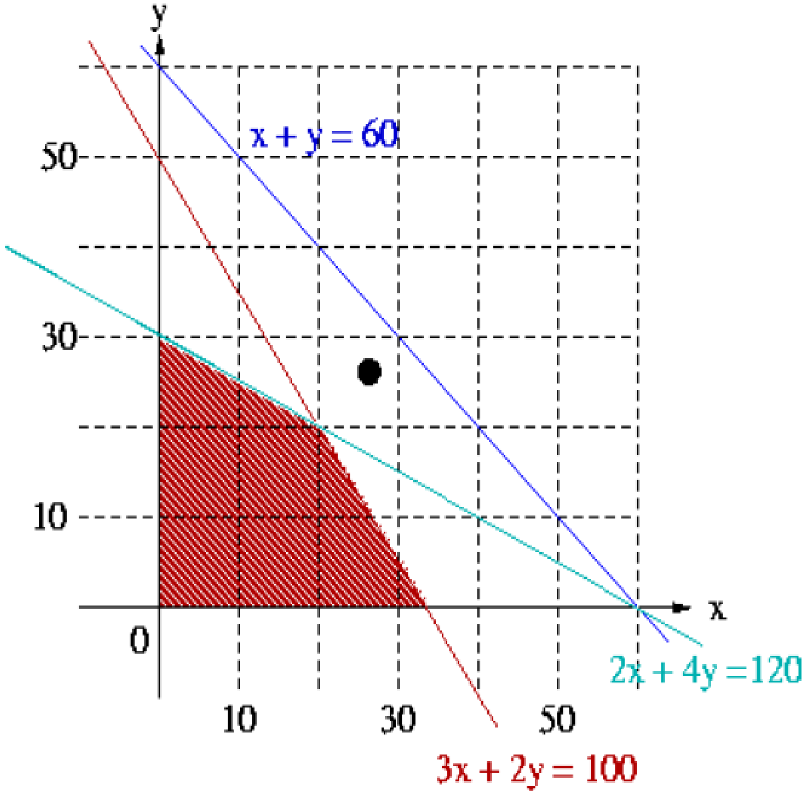
Bounded feasible region

Solutions



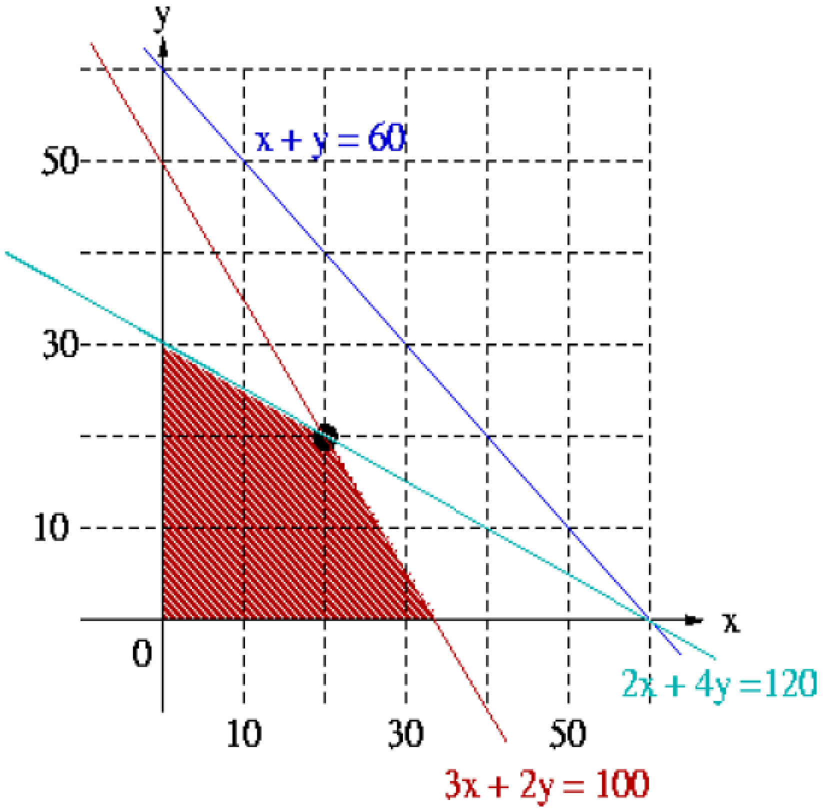
Feasible Solution

Solutions



Infeasible solution

Solutions

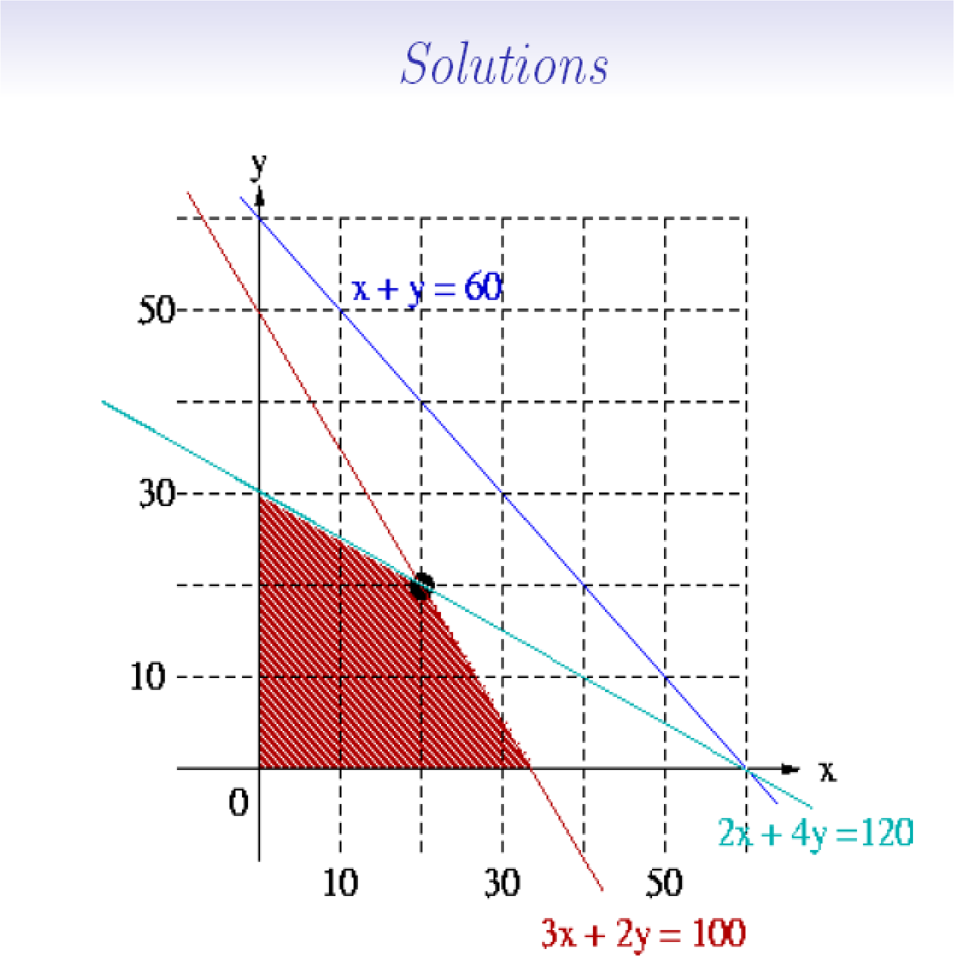


Optimal solution for $\max 200x + 300y$

x = 20

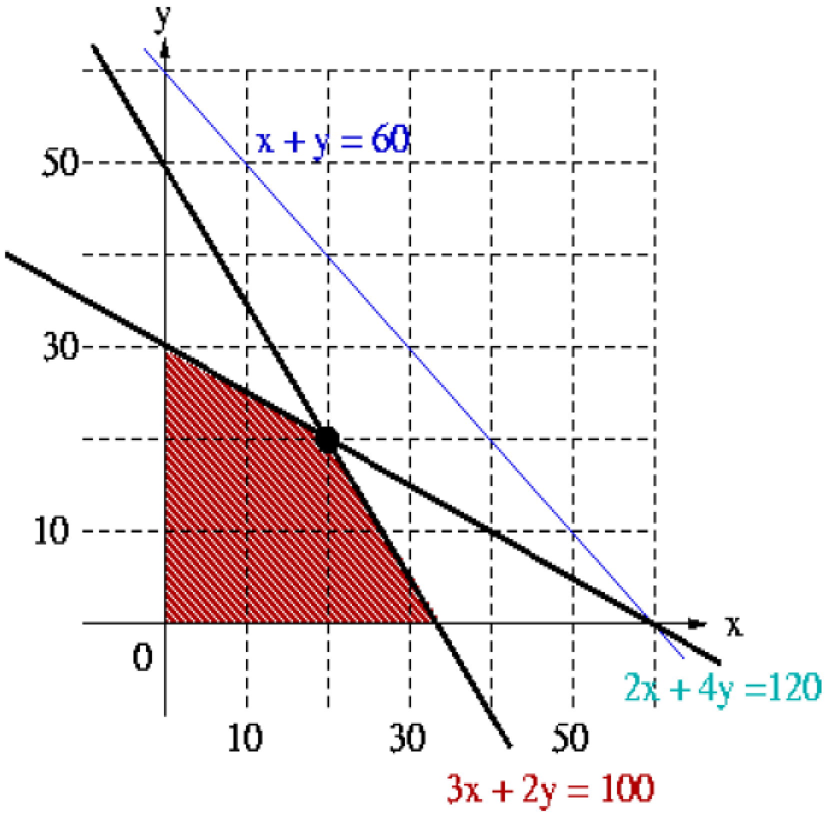
y = 20

Tight constraints



Optimal solution for max $200x + 300y$

Tight constraints



Max 200x + 300y

s.t. 3x + 2y ≤ 100 (labor)

2x + 4y ≤ 120 (fertilizer)

x + y ≤ 60 (acres)

x ≥ 0

y ≥ 0

Graphical method is trash

- Only really works in ~2 dimensions.
- In general: decision variables $x_1, x_2, \ldots, x_n$

$$
\begin{aligned}
\text{max/min} \quad & r_1x_1 + r_2x_2 + r_3x_3 + \cdots + r_nx_n \\
\text{subject to} \quad & a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n \leq u \\
& \qquad \vdots \\
& b_1x_1 + b_2x_2 + b_3x_3 + \cdots + b_nx_n \geq v \\
& \qquad \vdots \\
& c_1x_1 + c_2x_2 + c_3x_3 + \cdots + c_nx_n = w \\
& \qquad \vdots
\end{aligned}
$$

Solving LP

- **Theorem** [Dantzig 1947]:
  - Finite algorithm (Simplex method). Not polynomial, but fast in practice

- **Theorem** [Khachiyan 1979]:
  - LP can be solved in polynomial time in m, n, and L (Ellipsoid method).

. . .

Later: Karmarkar, Von Neumann etc.

Your turn

- Group up

- Create a Linear Program with:
  - At least 4 variables
  - At least 5 constraints

- The optimal value of your LP is your score, as long as it's a valid percentile between 0 and 100.

Modeling Power

- Linear constraints

  $3x + 4y \leq 120$

  $x_1 - x_2 = 0$

  $10x_1 + 20x_2 + 30x_3 \geq 150$

- Nonlinear constraints

  $3x + 4y < 120$

  $4xy = 10$

  $(3x+4y)/(x+y) \leq 3.5$ <- Can be linearized